Predicting the success of volatility targeting strategies:

Application to equities and other asset classes*


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24 March 2015 - Revised 5 June 2015

* We are grateful to Xiao Lu, Frank Moraux and Daniel Gabay for their insightful comments
Abstract:

Volatility targeting is a strategy that rebalances between a risky asset and cash in order to target a constant level of risk over time. When applied to equities and compared to a buy-and-hold strategy it is known to improve the Sharpe ratio and reduce drawdowns. We used Monte Carlo simulations based on a number of time-series parametric models from the GARCH family to analyze the relative importance of a number of effects in explaining those benefits. We found that volatility clustering and fat tails in return distributions are the two effects with the largest explanatory power. The results are even stronger when there is a negative relationship between return and volatility, which is known to be the case not only in equities but also to some extent in corporate bonds, government bonds and commodities. We used historical returns to simulate what the performance of a volatility targeting strategy would have been when applied to equities, corporate bonds, government bonds and commodities. The benefits of the strategy are more significant for equities and high-yield corporate bonds, which show the strongest volatility clustering, fat tails and negative relationship between returns and volatility. For government bonds and investment-grade bonds, which show less volatility clustering and a weaker negative relationship between returns and volatility, the benefits of the strategy were less marked.

Keywords: risk parity, constant volatility, asset allocation, risk budget, GARCH
Volatility targeting is a systematic strategy that invests in a risky asset and in the risk-free asset rebalancing the portfolio in such a way as to keep the ex-ante risk at a constant target level. This strategy is also known as constant risk, target risk or inverse risk weighting. The strategy made no sense if risky asset returns followed an independent and identically (i.i.d.) normal distribution. The volatility $\sigma$ of the risky asset returns would then be constant over time and could be estimated more accurately by increasing the history of returns used in its estimation. In such a case, allocating weight $w$ to assets in a portfolio or allocating risk budgets $w \times \sigma$ amounts to being two sides of the same coin. However, the returns of financial assets do not follow i.i.d. normal distributions and their volatility, which varies over time, is not even observable.

As noted by Mandelbrot [1963], financial asset returns show volatility clustering, which refers to the observation that large asset price fluctuations tend to be followed by large price fluctuations, either negative or positive, and small asset price fluctuations tend to be followed by small price fluctuations. Engle [1982] and Bollerslev [1986] introduced the ARCH (Auto-Regressive Conditional Heteroskedasticity) and GARCH (Generalized Auto-Regressive Conditional Heteroskedasticity) parametric models, respectively, which can be used to model asset returns taking into account volatility clustering. The survey of Poon and Granger [2003] lists 93 published and working papers that look at volatility forecasting and volatility modeling at different frequencies and for several asset classes. The survey of Andersen, Bollerslev, Christoffersen and Diebold [2005] also discusses volatility forecasting.

There is empirical evidence that the application of a volatility targeting strategy to equities targeting constant volatility over time does add value. Hoquard, Ng and Papageorgiou [2013] observe that not only do constant volatility portfolios deliver higher Sharpe ratios than buying and holding the underlying risky asset, but also that drawdowns are reduced. They show that targeting constant volatility helps to reduce tail risk. There is also theoretical support for the success of volatility-targeting strategies. Assuming constant mean return and varying volatility, Hallerbach [2012] proves that the volatility-targeting strategy improves the Sharpe ratio. The success of the strategy increases with better volatility forecasts. He shows that the higher the degree of volatility smoothing achieved by volatility weighting, the higher the risk-adjusted performance.

Hallerbach [2013] reviews different approaches to controlling risk in portfolios that make a clear distinction between managing risk in the cross-section and managing risk over time. He is of the view that volatility weighting over time does improve the risk-return trade off and suggests that, at least for equities, the success of volatility weighting also arises from a second effect known as the asymmetric volatility phenomenon: the fact that returns tend to be negatively correlated with volatility.
Cooper [2010], Kirby and Ostdiek [2012], Ilmanen and Kizer [2012] and Giese [2012] also defend the idea that the Sharpe ratio of portfolios of equities and cash targeting constant risk is higher than that of buying and holding equities.

In this paper, we investigate different GARCH models with parameters reflecting empirical observations and specific features of volatility. One of our goals is to establish the relationship between those parameters and the success of volatility-targeting strategies in increasing the Sharpe ratio. A second goal is to investigate the success of volatility targeting strategies when applied to other asset classes such as government bonds, corporate bonds or commodities. Research demonstrating the benefits of volatility targeting in portfolios has so far focused mostly on portfolios invested in equities and cash.

In the first part of the paper we discuss volatility targeting strategies and use Monte Carlo simulations to analyze the relative importance of the several possible effects behind the improved Sharpe ratio that was found when those strategies were applied to portfolios of equities and cash.

We use different stochastic models for returns and volatility to simulate constant volatility strategies. These different models allow us to consider the impact of volatility clustering, fat tails, the negative relationship between returns and volatility, and finally the inter-temporal diversification effect from equalizing the risk exposure at all times. We also look at the impact of the rebalancing frequency and the impact of the leverage.

We first consider i.i.d. normal distributed returns. In this case the volatility is constant and the strategy is thus obsolete. Nevertheless, its application using the historical standard deviation of returns as an estimator of volatility does not destroy value. The Sharpe ratio and drawdowns are similar to those from a buy-and-hold strategy, at least before transaction costs and market impact. Moving away from normal distributed returns, we then show that there is not just one but several effects explaining the increased Sharpe ratio in portfolios managed to target volatility, and we discuss their relative importance. Volatility clustering and the negative correlation between volatility and returns are indeed the two most important effects. But fat tails are also an important factor.

In the second part of the paper we look at the actual historical asset class return distributions. As demonstrate by Gatoumel and Ielpo [2012], the application of a hidden Markow model to asset classes’ returns shows two distinct volatility regimes, one with lower volatility and larger returns and one with higher volatility and lower returns. The results indeed suggest that the application of a volatility targeting strategy may successfully improve the Sharpe ratio, but the analysis is in-sample and thus not conclusive. The success of the strategy will depend on the success in forecasting volatility. Hence, we discuss the problem of forecasting volatility.
We find that GARCH models are good predictors of the future square of returns, $r^2$. However, when it comes to forecasting volatility, the short-term volatility models without assumptions for the long-term average volatility level appear to achieve a superior control of volatility ex-post and the most successful smoothing of risk as well as the larger improvement in the Sharpe ratio when compared to buy and hold. In particular, we show the superiority of the I-GARCH model in forecasting equity volatility and control for volatility ex-post. This observation is not limited to equities but also extends to the other asset classes considered here. This comparison is, however, limited to the GARCH models here considered. A number of other GARCH models have been proposed in literature. Nevertheless, we believe that the GARCH models we chose to investigate in here already capture the most important properties of volatility.

Using GARCH models we also show that the application of volatility targeting strategies for equities does result in an improvement in the Sharpe ratio and a reduction in portfolio drawdowns. The results are better for emerging equities than for developed equities. When applying the strategy to other asset classes we also find a large improvement in the Sharpe ratio of high-yield corporate bonds. However, the improvement of the Sharpe ratio is less marked for commodities, investment-grade corporate bonds and in particular for government bonds.

**IMPACT OF VOLATILITY PROPERTIES**

In this section, we discuss the construction of volatility targeting strategies and our approach to simulating the impact of different volatility properties on the risk-adjusted returns of such strategies.

**Volatility targeting strategy**

We consider a systematic strategy that invests in a risky asset and in the risk-free asset rebalancing of the portfolio in such a way as to keep the *ex-ante* risk at a constant target level. The weight of the risky asset in the portfolio is always positive and can be levered if necessary. The weight of the risk-free asset can be positive or negative, depending on whether the risky asset must be de-levered or levered in order to attain the constant target risk. Throughout the paper we use volatility as the risk measure except in the discussion of the optimality of inverse volatility weighting with inverse variance weighting.

This strategy is rebalanced on a daily basis, targeting a pre-defined level of volatility $\kappa$. The volatility of the underlying risky asset must be estimated every day. Given the current level of *ex-ante* volatility $\sigma_t$ of the risky asset and the pre-defined target volatility $\kappa$, the allocation to the risky asset, $w$, is simply $\kappa / \sigma_t$. The weight of cash is then $1 - \kappa / \sigma_t$ and the return of the strategy $r_{t,portfolio}$ is:
\[ \tau_{\text{portfolio}} = \tau_{\text{risky assets}} \frac{k}{\sigma_t} + \tau_{\text{rf}} \left(1 - \frac{k}{\sigma_t}\right) \quad (1) \]

with \( \tau_{\text{risky assets}} \) the return to the risky asset and \( \tau_{\text{rf}} \) the return to the risk-free asset, i.e. cash returns. By keeping the expected \textit{ex-ante} volatility of the two-asset portfolio constant over time we observe the performance of the strategy at the end of day \( t \) with the exposure of the strategy \( k/\sigma_t \) to the risky asset implemented at the end of the day \( t-1 \).

The portfolio is re-balanced at daily closing prices. Transaction costs were not considered. For cost efficiency, the strategy should be implemented using liquid investments such as equity index futures and money market instruments. Later we shall discuss approaches to significantly reduce turnover and thus transactions costs, which is of critical importance for the practical use of the strategy.

**Simulations based on stochastic models**

We used a number of stochastic models to analyze the impact of the different properties of the volatility of risky assets on the risk-adjusted returns of a volatility targeting strategy as defined in (1). Throughout the section we shall describe these models and the effects they capture in more detail. In these models the volatility \( \sigma_t \) is a function of past volatility and is used as an input in:

\[ r_t = \mu + \sigma_t z_t \quad \text{with} \quad z_t \sim \text{i.i.d.} \mathcal{N}(0,1) \quad (2) \]

with \( \mu \) being the expected return of the risky asset and \( z_t \) being a i.i.d. normal distributed noise variable.

We use GARCH models to simulate the impact of different properties of volatility, such as volatility clustering and fat tails, and extensions of the GARCH model to mimic the additional effects observed, such as higher frequency of fat tails or variable returns as a function of volatility.

Armed with a model for volatility \( \sigma_t \) and with equation (2) for the returns of the risky asset, we simulated 5 000 Monte Carlo scenarios of return time series each with 5 200 daily returns and analyzed the average behavior of the volatility targeting strategy in each of those simulations.

**Normal distributed returns and constant volatility**

In this first simulation we look at what happens if the risky asset returns were just i.i.d. normally distributed and demonstrate that if this is the case, with the volatility constant over time, then the application of the strategy using a rolling historical standard deviation of returns as an estimator of volatility neither adds nor destroys value. It simply leads to a higher average exposure to the risky asset over time.
In this first simulation the volatility in equation (2) is simply constant over time. Thus we set \( \sigma_t = 19.0\% \) for all \( t \) and we set \( \mu = 7.5\% \). The returns for the risky asset in excess of cash returns are then drawn from the i.i.d. normal distribution \( N (7.5\%, 19.0\%) \). For the volatility targeting strategy we set the target volatility \( \kappa = 19.0\% \) so that results can more easily be compared with the buy-and-hold strategy of the risky asset. For \( \tilde{\sigma}_t \) in (1) we use the standard deviation of simulated returns of the risky asset up to the time of rebalancing based on a 42-day rolling window. To analyze the average behavior of the volatility targeting strategy, we simulated 5 000 Monte Carlo scenarios each with 5 200 daily returns, i.e. 20 years, and using this normal distribution. The results of the Monte Carlo simulations are shown in the first column of Exhibit 1.

**Exhibit 1**: Buy-and-hold strategy compared with volatility targeting strategies with \( \kappa = 19.0\% \) rebalanced daily. In the first column, the risky asset returns in excess of cash were drawn from an i.i.d. normal distribution. In the other columns, the risky asset returns in excess of cash were generated from GARCH models with the parameters in the table. 5 000 Monte Carlo simulations of 5 200 daily returns each were used in the estimation of averages. In the first column, the volatility targeting volatility uses a 42-day historical volatility as an estimator of volatility. In the other columns the *ex-ante* volatility is based on the GARCH model employed.
If risky asset returns are drawn from an i.i.d. normal distribution then the volatility targeting strategy has the same Sharpe ratio as the buy-and-hold strategy in simulations. The improvement of the Sharpe ratio has a mean close to zero and a standard deviation of 2.5%. Interestingly, the average exposure to the risky asset is slightly higher than in the buy-and-hold strategy and thus the volatility, the excess return and the maximum drawdown are also slightly higher. This can be explained by the fact that the exposure to the risky asset is a function of $1/\sigma_t$, where $\sigma_t$ is the short-term volatility with a uniform distribution, i.e. there is the same probability of observing $\sigma = \sigma + \Delta$ as observing $\sigma = \sigma - \Delta$ for a given $\Delta$. Thus the average of $1/\sigma_t$ is higher than $\sigma$, which explains the larger exposure than in the buy-and-hold strategy.

If the risky asset returns are drawn from an i.i.d. normal distribution, then there is no added value from applying a volatility targeting strategy due to the underlying volatility being constant. But it is important to note that the application of the strategy would not deliver worse risk-adjusted returns than buy and hold.

Volatility clustering
The fact that the volatility of financial asset returns tends to show positive auto-correlation over several days is known as volatility clustering, meaning that high-volatility events tend to cluster over time. This property of volatility has been extensively discussed in academic literature (see for example Cont [2007]). Here, $\sigma_t$ follows a GARCH process as introduced by Bollerslev [1986]:

\[
\begin{align*}
  r_t &= \mu + \sigma_t z_t \quad \text{with} \quad z_t \sim \text{i.i.d.} \mathcal{N}(0,1) \\
  \sigma_t^2 &= \omega + \alpha (r_{t-1} - \mu)^2 + \beta \sigma_{t-1}^2
\end{align*}
\]

(3)

The larger the $\beta$ relative to $\alpha$ and the more stable the volatility, the larger the $\alpha$ and the more volatility clustering there is.

We repeated the simulations with the returns of the risky asset in excess of cash returns generated from model (3). In the second column of exhibit 1 we show the results of our simulations. The choice of parameters in the GARCH model was motivated by what can be estimated from a long history of S&P 500 returns using maximum likelihood approaches.

The average Sharpe ratio of a volatility targeting strategy simulation is higher than that from the simulation of a buy-and-hold strategy when volatility clustering is introduced in the time series of returns of the risky asset. The Sharpe ratio increases by 0.08 whereas the volatility of both strategies is comparable. Just as in the case of i.i.d. normally distributed returns, the volatility targeting strategy shows an average exposure to the risky asset in excess of 100%, but the effect is now even stronger than for the simulations based on i.i.d. normal distributed returns. 122.3% of the average exposure is required to generate the same level of volatility found in the buy-and-hold strategy. The maximum drawdown of the volatility targeting strategy is only slightly smaller than that for the buy-and-hold strategy.

With the expected return constant over time (i.e. $\mu$ is constant), the improvement in the Sharpe ratio can be explained by the volatility clustering that renders the volatility predictable. The volatility targeting increases the weight of the risky asset in the low volatility regimes and, with expected returns constant over time, when the Sharpe ratio for the risky asset is higher. In turn the volatility targeting strategy will reduce the weight of the risky asset in high volatility regimes when the Sharpe ratio is lower.

Intuitively, the improvement in the Sharpe ratio can be directly linked to the GARCH parameters $\alpha$ and $\beta$ in (3), which explain the clustering of volatility. Indeed, the more $\beta$ tends to one (note that in a GARCH model $\alpha + \beta$ must be inferior to one for the process to be stationary), the more the volatility will be stable over time and the less the clustering of volatility. Conversely, the larger that $\alpha$ is, the more the volatility clustering occurs.
Lamoureux and Lastrapes [1990] suggest that $\alpha + \beta$ is close to one due to the presence of shifts in the unconditional variance. If $\alpha + \beta$ is far from one (and below one), then the long-term variance $\omega$ plays a more important role in determining the volatility, and the distribution of returns converges towards an i.i.d. normal distribution.

**Exhibit 2:** Impact of $\alpha$ and $\beta$ on the Sharpe ratio of the volatility targeting strategy. Left chart: $\beta = 90\%$ and $\alpha$ varies from 0 to 9.5%. Right chart: $\alpha + \beta = 99.0\%$ and $\alpha$ varies from 59% to 1%. In both charts $\kappa = 19.0\%$, $\mu$ and $\omega$ were chosen so as to target 19.0% of volatility for the risky asset and the target Sharpe ratio = 0.39 (see parameters in exhibit 1). The target volatility $\kappa = 19.0\%$. The risky asset returns in excess of cash were generated from a GARCH model as in (3). 5 000 Monte Carlo simulations of 5 200 daily returns each were used.

In the left-hand chart in exhibit 2 we see a clear improvement of the Sharpe ratio of the volatility targeting strategy relative to buy and hold when $\alpha + \beta$ is closer to one. This means that the distribution of returns is far from i.i.d. normal and the parameter $\omega$ (the long-term volatility parameter) is low and has little impact. In the right-hand chart in exhibit 2 we see a clear improvement of the Sharpe ratio when $\alpha$ is large and $\beta$ is low while fixing $\alpha + \beta$ at a level close to one. Here, volatility clustering is important.

**Fat tails**

Another potentially important effect is the presence of fat tails in the distribution of returns of the risky asset. As mentioned by Cont [2001], for a large number of financial assets, the historical distribution of returns seems to exhibit power-law or Pareto-like tails, with a tail return that is finite and a power higher than two but lower than five. The exact shape of the tail is, however, difficult to verify. As mentioned by Moraux [2010], a standard Gaussian GARCH (1, 1) model already exhibits fat tails. Indeed, the unconditional kurtosis $k$ of the GARCH model depends on $\alpha$ and $\beta$:

$$ k = 3 + \frac{6\alpha^2}{1-(\alpha+\beta)^2-2\alpha^2} $$

(4)
One way to increase the frequency of fat tails in a GARCH model is to have the noise drawn from a t-Student distribution instead of white noise. This specification was proposed by Bollerslev [1987]. In this case, the returns of a risky asset follow:

\[
\begin{align*}
    r_t &= \mu + \sigma_t z_t \quad \text{with} \quad z_t \sim \text{i.i.d.} \mathcal{N}(0,1, \nu) \\
    \sigma_t^2 &= \omega + \alpha (r_{t-1} - \mu)^2 + \beta \sigma_{t-1}^2
\end{align*}
\]

(5)

with \( \nu \) the degree of freedom in the t-Student.

In the third column of exhibit 1 we compare the simulated buy-and-hold strategy against the simulated volatility targeting strategy when the frequency of fat tails is increased in the return distribution. Here, the volatility targeting strategy improves the Sharpe ratio, which is 0.12 higher than for the buy-and-hold strategy. The increase of the frequency of fat tails in the return distribution increases the average exposure to the risky asset to 132.6% for the same volatility of buy and hold, more than the 122.3% in exhibit 1 for the GARCH with a white noise. The model parameters were chosen in line with what would have been obtained from maximum likelihood estimation for the S&P 500 index in the long-term.

In exhibit 3 we show the improvement in the Sharpe ratio as a function of \( \nu \) in the t-Student distribution and find that the lower the \( \nu \), the higher the frequency of fat tails and the better the results. Extreme events are more important in high volatility regimes. The volatility targeting strategy reduces the weight of the risky asset in such regimes and thus appears quite successful at reducing the exposure to these events.

**Exhibit 3**: Impact of the degree of freedom \( \nu \) in the t-Student distribution on the improvement of the Sharpe ratio of a volatility targeting strategy relative to a buy and hold strategy. \( \beta = 90.0\% \), \( \alpha = 9.0\% \), \( \kappa = 19.0\% \) and the target Sharpe ratio is 0.39 (see parameters in exhibit 1). \( \omega \) and \( \mu \) are chosen to target this Sharpe ratio and a volatility of 19.0% for the risky asset. The risky asset returns in excess of cash were generated from a GARCH model with t-Student noise model as in (5). 5 000 Monte Carlo simulations of 5 200 daily returns each were used.

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**Exhibit 3**

![Graph showing improvement in Sharpe ratio vs. degree of freedom (\( \nu \))](chart.png)
The Sharpe ratio of the volatility targeting strategy is even higher when the distribution of returns of the risky asset not only shows volatility clustering but also has a higher frequency of fat tails. In periods of high volatility, the Sharpe ratio is even lower and the volatility even higher.

**Relationship between returns and volatility**

So far, the expected returns $\mu$ were kept constant over time, irrespective of the volatility regime. The increase in the Sharpe ratio in low-volatility regimes and its decrease in high-volatility regimes was solely due to changes in volatility.

A property often highlighted in the time series of risky asset prices is *gain and loss asymmetry*, i.e. the fact that the prices of financial assets usually increase slowly but tend to fall suddenly and fast. In other words, negative performances in asset prices induce higher volatility. This suggests that returns may show a negative correlation with volatility. One way to introduce *gain and loss* asymmetry in the GARCH models is to use the version proposed by Glosten, Jagannathan and Runkle [1993], known as the Glosten-Jagannathan-Runkle GARCH (GJR-GARCH model), which introduces a relationship between return and volatility:

\[
\begin{align*}
{r_t} &= \mu + \sigma_t z_t \quad \text{with} \quad z_t \sim \text{i.i.d. } N(0,1) \\
\sigma_t^2 &= \omega + \alpha I_{t-1} (r_{t-1} - \mu)^2 + \beta \sigma_{t-1}^2
\end{align*}
\]  

The parameter $\phi$ reflects the leverage effect and is usually estimated to be positive. In this case, negative returns increase future volatility by a larger amount than positive returns of the same magnitude. $I_{t-1}$ is 1 if $r_{t-1}$ is negative and 0 if positive.

We repeated the simulation using a GJR-GARCH model. As shown in the fourth column of exhibit 1, the simulations again show an improvement in the Sharpe ratio. However, assuming some form of negative relationship between the risky asset returns and volatility leads to a reduction in the maximum drawdown of the volatility targeting strategy when compared to the buy-and-hold strategy. Targeting a constant risk budget improves the Sharpe ratio – thanks to the clustering of volatility – and reduces the maximum drawdown due to a negative correlation between volatility and performance.

We can also introduce asymmetry in the volatility by imposing a conditional skewness effect. Hansen [1994], and also Campbell and Hentschel [1992], show how to incorporate skewness in GARCH processes. The Skewed-GARCH model is a GARCH model with noise generated from a skewed i.i.d. normal distribution and follows equation (3) but with $z$ replaced by an i.i.d. skewed normal distribution.
Another way to introduce a link between volatility and return is the GARCH-in-mean model proposed by Engle, Lilien and Robins [1987], which establishes a direct relationship between returns and volatility.

\[
\begin{align*}
    r_t &= \mu + \delta \sigma_t^2 + \sigma_t z_t \\
    \sigma_t^2 &= \omega + \alpha (z_{t-1} \sigma_{t-1})^2 + \beta \sigma_{t-1}^2
\end{align*}
\] (7)

Generally, the parameter $\delta$ is negative, which means that volatility and returns are negatively correlated. An increase in volatility will have a clear negative impact on the expected returns of the portfolio. The model parameters were chosen in line with the maximum likelihood estimation for the S&P 500 index in the long-term, indeed with $\delta$ negative.

Intuitively, the improvement in the Sharpe ratio can be directly linked to the parameter $\delta$ of the GARCH-in-mean model, which explains the impact of the volatility on the expected return of the strategy. And we have also a clear improvement of the ratio of the maximum drawdown to the volatility. The more the $\delta$ is negative, the lower (or even negative) the expected return following high volatility and the higher the Sharpe ratio for a volatility targeting strategy when compared to buy and hold.

**Exhibit 4:** Impact of $\delta$ on the Sharpe ratio of volatility targeting strategies. $\delta$ varies from -20 to 4. Target volatility was set to $\kappa = 19.0\%$. The risky asset returns in excess of cash were generated from a GARCH-in-mean model as in (7) with $\beta = 90.0\%$ and $\alpha = 9.0\%$. $\mu$ is chosen to target a Sharpe ratio of 0.39 and $\omega$ was chosen so that the annualized volatility is 19.0% (see parameters in exhibit 1). 5 000 Monte Carlo simulations of 5 200 daily returns each were used in the estimation of averages.

In exhibit 4, we see a clear and significant improvement in the Sharpe ratio of the volatility targeting strategy when compared to a buy-and-hold strategy when $\delta$ is negative and large. When $\delta$ is positive, indicating a positive relationship between volatility and expected returns, the Sharpe ratio deteriorates. This means that the more volatility is negatively correlated with expected returns, the stronger the improvement in the Sharpe ratio.
Inter-temporal diversification

Here we look at the improvement in risk-adjusted returns arising solely from better inter-temporal diversification, i.e. better allocation of risk over time. For this purpose we considered the following modified GARCH model:

\[
\begin{align*}
    r_t &= (\mu + z_t)\sigma_t \quad \text{with} \quad z_t \sim \text{i.i.d.} \mathcal{N}(0,1) \\
    \sigma_t^2 &= \omega + \alpha(r_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2
\end{align*}
\]  

(8)

This differs from the model in equation (3) in the description of the risky asset returns. The return is a function of the volatility and the $\mu$. The ratio of the unconditional mean divided by the unconditional variance is then constant over time. In this model, the larger the risk the larger the return, which makes sense from the point of view of financial theory. In this case, the risk-adjusted returns are constant over time and, as in the process described in (3), this process has volatility clustering with a constant Sharpe ratio.

As shown in the sixth column of exhibit 1, there is still a small improvement in the Sharpe ratio of the volatility targeting strategy arising from inter-temporal diversification. Investing with the same risk budget every day when compared to a buy-and-hold strategy does lead to a small improvement in the Sharpe ratio, even if it remains the same in higher and lower volatility regimes. A buy-and-hold strategy has a variable risk over time that is higher in higher volatility regimes. But when we compare the second column and the sixth column of exhibit 1 we find that this diversification effect is small.

Impact of rebalancing frequency and leverage

We shall now discuss the impact of the choice of the frequency of rebalancing. Aggregational gaussianity describes the fact that as one increases the time scale $t$ over which the returns of financial asset returns are calculated, their distribution increasingly resembles an i.i.d. normal distribution. The distribution of financial asset returns is not the same over different time scales. To test the importance of the frequency, we reduced the frequency of rebalancing in our back-tests from daily to semiannual.

In exhibit 5 we show the results of the simulations at different frequencies and compare them to a buy-and-hold strategy. When the frequency of rebalancing is decreased, the improvement in the Sharpe ratio is less marked as the risky asset returns between two rebalancing dates become more normally distributed. Rebalancing with lower frequency also results in less controlled risk, with ex-post volatility less constant over time than when rebalancing is more frequent. We can see this in exhibit 10 where the average annualized volatility calculated from daily returns is now at 21.6% for semiannual rebalancing when the target is 19.0%, whereas the daily rebalanced strategy achieves the target of 19.0% in ex-post. Even if the returns of the risky asset were i.i.d. normally distributed at all
time-scales, this problem can still be expected as the strategy will tend to be structurally more levered when rebalancing is less frequent, and thus the _ex-post_ risk is higher than expected. The results in exhibit 5 also show that rebalancing daily or weekly leads to very similar results. The _ex-post_ average annualized volatility, excess returns, and thus Sharpe ratio, are comparable. The maximum drawdown is also comparable. However, applying weekly frequency results in fewer transactions, and thus lower related costs, than rebalancing daily.

Our results have so far demonstrated that a volatility targeting strategy requires leverage and results in an average higher exposure to the risky asset than the buy-and-hold strategy. Obviously this poses a problem to the investor not allowed to take leverage or who is leverage-averse. The only way to still take advantage of the benefits of the strategy is to target a lower level of risk than that generated from a buy-and-hold strategy. The more the strategy is constrained, the smaller the improvement seen in the Sharpe ratio. At \( \kappa = 19.0\% \) and capped at 100\%, the strategy shows an improvement of the Sharpe ratio of only 0.046. Without a constraint on leverage the Sharpe ratio would be higher at 0.08.

In exhibit 5, we show the results of simulations with the exposure to equities capped at 200\% and 100\%. 200\% exposure to the risky asset is not often reached at \( \kappa = 19.0\% \) and thus, a cap of the exposure at a maximum of 200\% has little impact on the strategy. If we impose a no-leverage constraint, i.e. exposure capped at 100\% and still target risk \( \kappa = 19.0\% \), then we find that _ex-post_ the volatility is only 15.8\% and the improvement in the Sharpe ratio is small, with the average exposure to the risky asset at 93.5\%. Simulations in this part have been done without capped exposure but we see that a constraint at 200\% for practical reason has a small impact on results.

**Exhibit 5**: Buy-and-hold strategy compared with a volatility targeting strategy with \( \kappa = 19.0\% \). The risky asset returns in excess of cash were generated from a GARCH model as in (3) with the same choice of parameters in the second column of exhibit 1, but the strategy is rebalanced at different frequencies and constraints on exposure are applied. 5 000 Monte Carlo simulations of 5 200 daily returns each were used in the estimation of averages.
IMPLEMENTATION OF VOLATILITY TARGETING STRATEGIES

We have learned from the simulations based on GARCH models that rebalancing portfolios of a risky asset and cash to target a constant risk budget over time adds value when compared to a buy-and-hold strategy. The first improvement is in the Sharpe ratio. The more the volatility of the risky asset shows clustering, the larger the improvement in the Sharpe ratio. The improvement is even larger when there are fat tails in its distribution of returns. The second improvement is a smaller maximum drawdown, in particular if the distribution of returns has lower average returns in high volatility periods and higher average returns in lower volatility periods, i.e. when returns and volatility are negatively correlated. There is also a positive, although small, impact from inter-temporal diversification. Rebalancing frequently is important but comes with high turnover. Weekly rebalancing delivers most benefits of the strategy with a much lower implementation cost than daily rebalancing because much less trading is required.

We now focus on the different properties of the volatility of returns not only on equities but also on a number of other asset classes. We discuss efficient ways of implementing the strategy and analyze the results from historical back-tests.

Historical data

All data was downloaded from Bloomberg. We used time series of total returns in USD with dividends reinvested in the case of equities, coupons reinvested and interest accrued in the case of bonds. Data runs from 1 January 1988 through 31 December 2013, except for the S&P 500 and the risk free rate, for which the historical data runs from 1 January 1980 through 31 December 2013.

The S&P 500 (ticker SPTR Index) measures the performance of 500 capitalization-weighted large-cap US stocks. The Russell 1000 (ticker RU10INTR Index) measures the performance of capitalization-weighted large-cap US stocks. The MSCI Emerging Market Index (ticker NDUEEGF Index) measures the performance of capitalization-weighted stocks in emerging markets. The index is denominated in USD and the currency risk is not hedged. The S&P Commodities Index (ticker SPGCCITR Index) measures the performance of commodities. The US High Yield Index (ticker H100 Index), US Corporate Investment Grade Index (ticker C0A0 Index) and US 10-year Government Bonds Index (built from tickers USGG07YR Index and USGG10YR Index targeting a constant duration of 10 years and ignoring convexity) measure the performance of US high-yield and investment-grade corporate bonds and bonds issued by the US government, respectively. We use the 3-month US Dollar LIBOR as a proxy for the risk-free rate (ticker US0003M Index).

Forecasting volatility
Perfect volatility smoothing would require perfect volatility foresight. However, that is impossible because the volatility is not even observable. Still, volatility forecasting has been attracting much academic and practitioner attention for more than three decades (e.g. Poon and Granger [2003]). Two well-documented approaches to forecasting volatility are times-series models and implied-volatility methods. There are also non-parametric approaches that make few or no assumptions, such as the historical standard deviation, but these perform poorly compared to times-series models, according to Pagan and Schwert [1990]. Neural networks can also be used to compute risk. These methods would be classified in the category of machine-learning approaches but we do not consider them here.

Implied volatility approaches offer a number of challenges, the first being the large choice of implied volatility measures available arising from different strike and maturities options. There are also liquidity issues as discussed by Mayhew [1995] and Hentschel [2003]. In addition, implied volatility may be overpaid as investors are interested in the potential payoff or because they fear for the returns of a portfolio and are prepared to overpay for insurance via put options. These two reasons should lead to higher-than-fair prices, as discussed by Figlewski [1997].

The most popular approach to forecasting volatility is times-series modeling. One reason is because such modeling only requires easily accessible historical information. These modeling approaches start with a parametric model for the returns of financial assets. First, the features required are defined and then an actual financial distribution is fitted. If successful then it is reasonable to expect that the parametric model should have forecasting power. Typical key expected features are volatility clustering and volatility asymmetry.

We focus our attention on four such models. The first is the GARCH model already discussed as in equation (3). The second, already introduced in equation (6), is the GJR-GARCH. The third, already introduced in equation (7), is the GARCH-in-mean. The fourth, already introduced in equation (5), is the GARCH with t-student noise. Finally, the fifth is the I-GARCH (Integrated-GARCH), which is the special case of a GARCH (1, 1) and was introduced by Engle and Bollerslev [1986].

The I-GARCH is defined from equation (3) by setting the long-term average volatility \( \omega = 0 \) and \( \alpha + \beta = 1 \). This simplification of the GARCH model tends to work well for forecasting volatility in practical terms because the long-term volatility of financial assets \( \omega \) is difficult to estimate and may not even be stationary. However, the model cannot be used in simulations like those performed in the first part of this paper because \( \sigma_t \) and \( r_t \) would converge to 0.

Of the many extensions of GARCH models that have been proposed to mimic the additional effects observed in financial markets, switching GARCH models have gained attention in recent years. In these models, the parameters of the volatility can change over time according to a latent (i.e.
unobservable) variable taking values in the discrete space \(\{1, \ldots, K\}\) where \(K\) is an integer defining the number of regimes or states. Changes in returns according to the regime of volatility can thus be considered, i.e. the idea that changes in the unconditional variance have an impact on returns (see Lamoureux and Lastrapes [1990]). The empirical and fundamental motivation for such models is given by Friedman and Laibson [1989] and Ederington and Lee [2001] and details of the implementation of both ARCH and GARCH models with regime-switching models is given by Hamilton [1989], Gray [1996] and Klaassen [2002]. But the robust estimation of the parameters of these models is difficult, so it is difficult to use them in forecasting. The convergence of parameters is slow and 20 years of data is usually not sufficient for a significant estimation of parameters.

**Historical simulations with forecast volatility**

We performed historical simulations to assess which of the five GARCH models described would have generated superior volatility forecasts. To do so, we calculated the Mean Square Error (MSE), which looks at the average of the square difference between square returns in \(t\) and estimation of the variance in \(t\). And we also look at the Mean Absolute Error (MAE), which measures the average between the absolute spread instead of the square difference. We calibrated the model parameters as suggested by Hocquard, Ng and Papageorgiou [2013] using an expanding window and a maximum likelihood approach. The first ten years are used for the first calibration. The error is back-tested in the eleventh year while keeping the model parameters constant. The model was then re-estimated with 11 years of data and used to back-test the twelfth year, and so on. The last re-estimation uses data between 1 January 1980 and 31 December 2012 for the error in 2013. We show the results of such estimation in exhibit 6.

**Exhibit 6:** Two measures of mean forecasting error for the five proposed volatility models.

<table>
<thead>
<tr>
<th>Model</th>
<th>MSE</th>
<th>MAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>GARCH</td>
<td>1.55E-07</td>
<td>1.41E-04</td>
</tr>
<tr>
<td>GARCH with t-student noise</td>
<td>1.56E-07</td>
<td>1.41E-04</td>
</tr>
<tr>
<td>GJR-GARCH with white noise</td>
<td>1.49E-07</td>
<td>1.38E-04</td>
</tr>
<tr>
<td>GARCH in mean with white noise</td>
<td>1.55E-07</td>
<td>1.42E-04</td>
</tr>
<tr>
<td>I-GARCH</td>
<td>1.58E-07</td>
<td>1.43E-04</td>
</tr>
</tbody>
</table>

The GJR-GARCH model has the lowest average forecasting errors, i.e. delivers the most accurate forecast of \(r^2_t\). However, the differences in mean forecasting errors in exhibit 6 are too small to be conclusive and, as shown in the Annex B, these differences are not significant. Hence, we cannot really tell which of these models is the most accurate.

Practitioners tend to use historical volatility as a risk measure. Thus, we also used a different test based on comparing *ex-ante* target volatility with *ex-post* volatility. We calculated the three-year
rolling *ex-post* volatility of the returns of volatility targeting strategies every day and checked which of these models achieved the best control of volatility in *ex-post*. Indeed, this approach is more in line with industry practice. In the example, the target volatility is \( \kappa = 10\% \) and the results are for the S&P 500. We calibrated the model parameters as before for the MSE and MAE. The portfolio is rebalanced between the risky asset and the risk-free asset every day in order to target a constant risk level as in (1). We show the results of such simulations in exhibit 7.

**Exhibit 7:** Three-year rolling *ex-post* volatility for volatility targeting strategies applied to the S&P 500. The target volatility is \( \kappa = 10\% \) and the forecast volatility is based on four different GARCH models with parameters estimated from an expanding window once every year at the start of each year.
The I-GARCH model clearly shows superior control of the volatility of the portfolio in these historical simulations. The three-year rolling volatility deviates less from the target volatility $\kappa$ when the I-GARCH model is used. In the other models, the ex-post volatility falls well below target between 1991 and 1996 and again between 2003 and 2008. This fall in volatility is mainly due to the long-term volatility parameter $\omega$, which is the most difficult to estimate.

We now focus on the performance of volatility targeting strategies in this historical period. The impact of transaction costs is not taken into account. We use the 3-month US Dollar LIBOR as the proxy for the risk-free rate. The results are presented in exhibit 8 and were calculated from 1 January 1990 through 31 December 2013, as before. When compared to buying and holding the S&P 500, the volatility targeting strategy improves the Sharpe ratio and reduces the maximum drawdown, irrespective of the GARCH model used. The S&P 500 shows volatility clustering and some short-term serial correlation in the returns. The Student’s t-test is significant with a confidence level of 69% (the statistic of the t-distribution with degrees of freedom 21, i.e. number of years minus 1, equal 0.50). The volatility of the volatility is the lowest when the I-GARCH is used, confirming its superior volatility forecasting accuracy. The highest Sharpe ratio and the smallest drawdown are also generated by the volatility targeting strategy based on the I-GARCH.

**Exhibit 8:** Comparison of a buy-and-hold strategy for the S&P 500 with volatility targeting strategies targeting $\kappa = 10\%$ volatility and using forecasted volatility from five different GARCH models. The GARCH model parameters are estimated from an expanding window once every year at the start of each year.
Exhibit 8 could raise some concerns. The first is the relatively low significance of the Student’s t-test at only 60% to 75%. Should we have not carried out the analysis in the first part of this paper based on Monte Carlo simulations of GARCH models, this would have been a concern. The strong backing from those simulations combined with the consistency of results thus shows that the strategy provides benefits in the long term and that the period from 1990 to 2013 is not sufficiently long for a strong statistical validation. This is perhaps not surprising since we know that the number of transitions between high and low volatility regimes and the number of fat tail events in the 22-year period are not likely to be sufficient for statistical validation. However, the results do point in the same direction as the Monte Carlo simulations.

The second concern is the small difference between GARCH models. But as expected, small differences in the MSE or MSA produce small differences in the improvement of the Sharpe ratio when compared to the buy-and-hold strategy. The improvement of Sharpe ratio falls in the range of +0.08 to +0.12.

The third concern is the high turnover from a daily implementation of the volatility targeting strategy. But as we showed in exhibit 5, the benefits of the strategy can be found even at a weekly frequency. Reducing the frequency from daily to weekly already reduces the turnover by 5, i.e. to acceptable levels from a practical point of view. An even more efficient approach to reduce turnover is to monitor the volatility daily but only change the portfolio allocation when volatility rises or falls significantly. We can define a corridor of volatility about the target volatility \( \kappa \), with an upper threshold at \( \kappa + \Delta \) and a lower threshold at \( \kappa - \Delta \), and rebalance only when the volatility falls outside the corridor. The size of \( \Delta \) is determined from the compromise between the benefits of the volatility targeting strategy in terms of improvement of the Sharpe ratio and reduction of drawdowns and the cost of implementing it.

**Impact of \( \alpha \) and \( \beta \) for different asset classes**
We will now focus on the application of the volatility targeting strategy to other equity indices and asset classes.

In exhibit 9 we show the parameters of a GARCH model with t-Student noise as in (5) estimated from the historical returns of the indices using maximum likelihood approaches over the period indicated. The first observation is that in all cases $\alpha + \beta$ is close to one, indicating that all the time series of returns are indeed stationary, as expected. High-yield corporate bonds have the largest $\alpha$. Emerging market equities also have a large $\alpha$. Large-cap US equities have larger $\alpha$ than commodities. Investment-grade corporate bonds and government bonds have the smallest $\alpha$. From these results we expect a volatility targeting strategy to provide the largest improvement in Sharpe ratio and largest reduction of drawdowns for high-yield corporate bonds and for emerging market equities, where volatility clustering is the strongest. The smallest improvement is expected for government bonds and for investment-grade corporate bonds, where volatility clustering is the weakest.

**Exhibit 9:** Parameters of a GARCH with t-Student noise model as in equation (5) estimated from historical returns for different asset classes from 1 January 1993 to 31 December 2013.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>1.6E-06</td>
<td>7.0E-07</td>
<td>8.0E-07</td>
<td>3.0E-07</td>
<td>2.0E-07</td>
<td>1.0E-07</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>9.6%</td>
<td>6.1%</td>
<td>5.4%</td>
<td>4.3%</td>
<td>21.7%</td>
<td>4.0%</td>
</tr>
<tr>
<td>(t-stat of $\alpha$)</td>
<td>11.40</td>
<td>9.90</td>
<td>11.00</td>
<td>8.7</td>
<td>12.70</td>
<td>8.7</td>
</tr>
<tr>
<td>$\beta$</td>
<td>89.3%</td>
<td>93.3%</td>
<td>94.1%</td>
<td>94.6%</td>
<td>75.7%</td>
<td>95.0%</td>
</tr>
<tr>
<td>(t-stat of $\beta$)</td>
<td>89.5</td>
<td>132.3</td>
<td>171.9</td>
<td>136.7</td>
<td>38.0</td>
<td>162.1</td>
</tr>
<tr>
<td>$\alpha + \beta$</td>
<td>98.9%</td>
<td>99.4%</td>
<td>99.5%</td>
<td>98.9%</td>
<td>97.4%</td>
<td>99.0%</td>
</tr>
<tr>
<td>t-Student</td>
<td>7.1</td>
<td>5.6</td>
<td>7.4</td>
<td>7.6</td>
<td>3.7</td>
<td>6.7</td>
</tr>
<tr>
<td>(t-stat of t-Student)</td>
<td>13.2</td>
<td>16.8</td>
<td>12.7</td>
<td>12.7</td>
<td>33.9</td>
<td>12.8</td>
</tr>
</tbody>
</table>

We also performed historical simulations of volatility targeting strategies applied to each of these asset classes in much the same way as we did for the S&P 500. We used different forecasted volatility models. Due to the shorter history we reduced the starting window of estimation of the parameters of the GARCH models to five years instead of ten. The target volatility is $\kappa = 5\%$. The results of the capability of volatility forecast are shown in exhibit 10. The first concern is the forecast of $\hat{\sigma}^2$. We have different results for different asset classes. The best forecast errors are not obtained with the same model for all assets classes. In addition, we observe small differences between these models. The second concern is the volatility of three-year rolling volatility. The I-GARCH model delivers the best results for all asset classes. The volatility of ex-post volatility of the strategy is the smallest with this model. As for the S&P 500, the I-GARCH achieves a better control of the volatility over time.
**Exhibit 10:** Comparison of the two means forecasting error for the five proposed models for different asset classes and comparison of volatility (standard deviation) of the three-year rolling ex-post volatility for the same asset classes and the same volatility models. The target volatility is $\kappa = 5\%$.

The GARCH models parameters are estimated from an expanding window once every year at the start of each year. The period is 1 January 1988 through 31 December 2013.

<table>
<thead>
<tr>
<th>MSCI Emerging Markets</th>
<th>Volatility-targeting strategy</th>
<th>GARCH</th>
<th>GARCH with t-student noise</th>
<th>GJR-GARCH with white noise</th>
<th>GARCH in mean with white noise</th>
<th>I-GARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSE</td>
<td></td>
<td>1.60E-07</td>
<td>1.60E-07</td>
<td>1.57E-07</td>
<td>1.62E-07</td>
<td>1.63E-07</td>
</tr>
<tr>
<td>MAE</td>
<td></td>
<td>1.50E-04</td>
<td>1.50E-04</td>
<td>1.50E-04</td>
<td>1.53E-04</td>
<td>1.54E-04</td>
</tr>
<tr>
<td>Volatility of 3-year rolling volatility</td>
<td></td>
<td>0.27%</td>
<td>0.29%</td>
<td>0.29%</td>
<td>0.36%</td>
<td>0.12%</td>
</tr>
<tr>
<td>Russell 1000</td>
<td></td>
<td>1.75E-07</td>
<td>1.78E-07</td>
<td>1.70E-07</td>
<td>1.79E-07</td>
<td>1.84E-07</td>
</tr>
<tr>
<td>MAE</td>
<td></td>
<td>1.48E-04</td>
<td>1.51E-04</td>
<td>1.44E-04</td>
<td>1.57E-04</td>
<td>1.54E-04</td>
</tr>
<tr>
<td>Volatility of 3-year rolling volatility</td>
<td></td>
<td>0.53%</td>
<td>0.24%</td>
<td>0.58%</td>
<td>0.51%</td>
<td>0.20%</td>
</tr>
<tr>
<td>S&amp;P GSCI Commodity</td>
<td></td>
<td>1.54E-07</td>
<td>1.54E-07</td>
<td>1.55E-07</td>
<td>1.54E-07</td>
<td>1.54E-07</td>
</tr>
<tr>
<td>MAE</td>
<td></td>
<td>1.96E-04</td>
<td>1.95E-04</td>
<td>1.95E-04</td>
<td>1.98E-04</td>
<td>1.98E-04</td>
</tr>
<tr>
<td>Volatility of 3-year rolling volatility</td>
<td></td>
<td>0.13%</td>
<td>0.14%</td>
<td>0.14%</td>
<td>0.20%</td>
<td>0.06%</td>
</tr>
<tr>
<td>US 10Y Government Bonds</td>
<td></td>
<td>2.72E-09</td>
<td>2.70E-09</td>
<td>2.72E-09</td>
<td>2.70E-09</td>
<td>2.71E-09</td>
</tr>
<tr>
<td>MAE</td>
<td></td>
<td>2.50E-05</td>
<td>2.51E-05</td>
<td>2.51E-05</td>
<td>2.59E-05</td>
<td>2.59E-05</td>
</tr>
<tr>
<td>Volatility of 3-year rolling volatility</td>
<td></td>
<td>0.31%</td>
<td>0.25%</td>
<td>0.32%</td>
<td>0.31%</td>
<td>0.12%</td>
</tr>
<tr>
<td>US High Yield Bonds</td>
<td></td>
<td>1.97E-09</td>
<td>1.90E-09</td>
<td>1.95E-09</td>
<td>1.98E-09</td>
<td>2.08E-09</td>
</tr>
<tr>
<td>MAE</td>
<td></td>
<td>8.61E-06</td>
<td>8.41E-06</td>
<td>8.88E-06</td>
<td>8.80E-06</td>
<td>9.11E-06</td>
</tr>
<tr>
<td>Volatility of 3-year rolling volatility</td>
<td></td>
<td>0.54%</td>
<td>0.61%</td>
<td>0.53%</td>
<td>0.52%</td>
<td>0.23%</td>
</tr>
<tr>
<td>US Investment Grade Bonds</td>
<td></td>
<td>1.02E-05</td>
<td>1.02E-05</td>
<td>1.02E-05</td>
<td>1.05E-05</td>
<td>1.06E-05</td>
</tr>
<tr>
<td>Volatility of 3-year rolling volatility</td>
<td></td>
<td>0.30%</td>
<td>0.26%</td>
<td>0.30%</td>
<td>0.22%</td>
<td>0.12%</td>
</tr>
</tbody>
</table>

**Exhibit 11:** Comparison of the historical simulations of a buy-and-hold strategy for different asset classes with a volatility targeting strategy with target volatility $\kappa = 5\%$ and using different forecast volatility models. The GARCH model parameters are estimated from an expanding window once every year at the start of each year. The period is 1 January 1988 through 31 December 2013.
We now consider the historical simulation of the volatility targeting strategy applied to these different asset classes. We use the same methodology and same parameters as before for the measure of the forecast error. The results of the simulations and comparison with the buy-and-hold strategy are shown in exhibit 11. The best improvement in the Sharpe ratio was found for high-yield corporate bonds, for which volatility clustering is the strongest. For equities, the improvement is also large, and even more so for emerging equities, as expected from the GARCH parameters. For commodities, the benefit is small and for US investment-grade and government bonds it is negligible. The Student’s t-test shows much higher significance for the high-yield corporate bonds’ improvement of Sharpe ratio at about 99% significance level. This is because the effects are much stronger for this asset class than for the others and thus a shorter period is sufficient to statistically confirm the benefits of the strategy.

For some asset classes, the GJR-GARCH model delivers a better improvement of the Sharpe ratio when compared to the I-GARCH model, in particular for US high-yield bonds. The GJR-GARCH
model simply fits history better than the I-GARCH model. But when it comes to delivering a constant volatility close to target over time, the I-GARCH model was proven to be superior.

CONCLUSIONS

We confirm that in a world where the distribution of risky asset returns is not i.i.d. normal, volatility targeting strategies that rebalance between the risky asset and the risk-free rate to keep volatility constant over time can add value when compared to buy-and-hold strategies increasing the Sharpe ratio and reducing the size of drawdowns. We show that the key effects behind these benefits are volatility clustering and fat tails. A negative correlation between returns and volatility increases these benefits. Assets with stronger volatility clustering and fat tails, e.g. high-yield bonds, show the most significant improvements. The effects are also significant for equities but less so for commodities. For investment-grade corporate bonds and government bonds, volatility clustering has not been sufficiently strong in the last 20 years to generate any significant or visible effects.

Our Monte Carlo simulations based on GARCH models allow us to confirm these effects and to analyze in detail the dependence of the benefits of volatility targeting strategies on the parameters of the models. The volatility clustering effect is in essence a market-timing effect. If the volatility changes and returns remain constant then the Sharpe ratio is higher in lower volatility regimes and increasing the weight of the risky asset in such periods will result in better risk-adjusted performances. When fat tails are present in the distribution of the returns of the risky asset, reducing the exposure to the risky asset in regimes of higher volatility, and when fat tails are larger, results in smaller drawdowns than when following a buy-and-hold strategy. The effects are more pronounced if the distribution of risky asset returns also shows a smaller mean return in regimes of higher volatility and a larger mean return in regimes of lower volatility.

The strategy is resilient to changing the frequency of rebalancing of the portfolio and the benefits are seen irrespective of whether daily or weekly rebalancing is performed. Reducing the frequency further erodes some of the benefits and increases ex-post volatility. Levels of acceptable turnover for a practical implementation can be found with weekly rebalancing. Further reduction of turnover can be achieved with daily monitoring of volatility and rebalancing only when the volatility changes significantly. In addition, introducing a cap on the exposure to the risky asset at 200% does not reduce the improvement of the Sharpe ratio.

We recommend the use of I-GARCH models for the practical implementation of the strategy. This shows the strongest predictive power when it comes to keeping ex-post volatility reasonably close to target. The improvement in the Sharpe ratio when compared to buy-and-hold strategies was often larger than that obtained using other GARCH models.
NOTES

1- These parameters are comparable to those found for the S&P 500 in the long-term

2- The volatility is calculated using the estimator $\hat{\sigma}_t = \sqrt{\frac{1}{42} \sum_{i=t-42}^{t-1} r_i^2}$. The choice of 42 days corresponds to the estimation of 2 months daily volatility, which is typically used by practitioners in short-term volatility models

ANNEX A

In finance, General Auto Regressive Conditional Heteroskedasticity (GARCH) models are used to model the variance of observed returns whenever there is a reason to believe that the variance of past returns will have an impact on the variance of future returns.

\[ r_t = \mu + \sigma_t z_t \text{ with } z_t \sim \text{i.i.d. } N(0,1), \]
\[ \sigma^2 = \omega + \alpha (r_{t-1} - \mu)^2 + \beta \sigma_{t-1}^2. \]  

(9)

Let us denote $\varepsilon_t$ the returns residual in equation (9) relative to the mean of the process. The process is thus split into a stochastic piece $z$ and a time-dependent standard deviation $\sigma$. The conditional mean of the process is $E(\varepsilon_t | F_t) = 0$ where $F_t$ is the filtration defined with the past value of $\varepsilon_t$ of $\varepsilon_t^2$ and of $\sigma^2$. The conditional variance is $V(\varepsilon_t | F_t) = \sigma^2$. The two first unconditional moments can be computed by iteration with the conditional moments of GARCH model. They can be computed as $\sigma_t^2$ is $F_t$ quantified. The unconditional mean process is $E(\varepsilon_t) = E(E(\varepsilon_t | F_t)) = 0$. The unconditional variance is defined by iteration, $V(\varepsilon_t) = E(\sigma^2)$. For $n$ iterations and with the condition of $|\alpha + \beta| < 1$, the unconditional variance of the returns is finite and is written as follow:

\[ E[\varepsilon_t^2] = \frac{\omega}{1-(\alpha + \beta)} \]  

(10)

The variance is positive if $\omega$ is the same sign of $1 - (\alpha + \beta)$.

Given a sample $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of $n$ i.i.d. observations which comes from a distribution $f(\varepsilon)$ with unknown parameters $\theta$ then the joint density function is:

\[ f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n | \theta) = f(\varepsilon_1 | \theta) \ast f(\varepsilon_2 | \theta) \ast \ldots \ast f(\varepsilon_n | \theta) \]  

(11)

The likelihood function changes the point of view, compared to (11), by considering the observed values $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ to be fixed parameters of the function whereas $\theta$ will be the function’s variable and allowed to vary freely. The likelihood function is written:
\( \mathcal{L}(\theta|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n|\theta) = \prod_{i=1}^{n} f(\varepsilon_i|\theta) \) \hspace{1cm} (12)

In practice, it is often more convenient to work with the logarithm of the likelihood function, called the log-likelihood:

\[ \ln \mathcal{L}(\theta|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \sum_{i=1}^{n} \ln f(\varepsilon_i|\theta). \] \hspace{1cm} (13)

An estimate of the value of \( \theta \) can then be obtained by maximizing equation (13) using numerical optimization.

**ANNEX B**

We used the Student’s t-test to assess whether the forecasting power of one model is larger than that of another. According to this test, the difference between models are not significant.

**Exhibit 12:** Comparison of the mean forecasting errors for the five proposed volatility models and the Student’s t-test to assess whether the forecasting errors are different from one model to another.

<table>
<thead>
<tr>
<th></th>
<th>GARCH</th>
<th>GARCH with t-student noise</th>
<th>GJR-GARCH with white noise</th>
<th>GARCH in mean with white noise</th>
<th>I-GARCH</th>
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<td><strong>MSE</strong></td>
<td>1.55E-07</td>
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<td>1.49E-07</td>
<td>1.55E-07</td>
<td>1.58E-07</td>
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<td>-19.0%</td>
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<td>17.9%</td>
<td>21.0%</td>
<td>-22.6%</td>
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<tr>
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<td>-17.5%</td>
<td>-</td>
<td>9.1%</td>
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<tr>
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<td>-4.3%</td>
<td>-27.2%</td>
<td>-9.2%</td>
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</tr>
<tr>
<td><strong>MAE</strong></td>
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<td>1.41E-04</td>
<td>1.38E-04</td>
<td>1.42E-04</td>
<td>1.43E-04</td>
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</tr>
</tbody>
</table>

**REFERENCES**


